Computational Geometry
Exercise 2

Due: 3/22/2000

1 Exercise 2

1.1 Arrangements

(25 points)

Planar arrangements are usually represented by a quad-edge data-structure that enable one to traverse the arrangement. Operations include: (i) Get any face in the arrangement, (ii) Get any edge in the arrangement, (iii) Get next edge around face in a counter-clockwise direction, (iv) Get the same edge directed in the other direction (i.e., we consider the edge to be directed in counterclockwise direction around their faces. Each edge corresponds to two such edges. (v) Get next edge having the same source point as the current edge, and in counterclockwise order around the source. (vi) Get the face adjacent to the current edge to its left.

Describe a data-structure for representing an arrangements, so that all those operations are supported efficiently (add other operations that might be useful). Show how one can compute the arrangement of n lines in $O(n^2 \log n)$ time. (remark: this can be done in $O(n^2)$ but requires a considerably more careful implementation, and using the zone theorem).

1.2 Incidences in the Plane

1. (10 points) Prove that $\sum_{i=1}^{n} i\phi(i) = \Omega(n^3)$.

2. (15 points) Prove that the number of incidences between a set of m points and n lines is $\Omega(n^{2/3}m^{2/3} + m + n)$.

   Hint: Let $g(i,j,s,r)$ denote the line passing through the points with coordinates (i,j) and $(i+s, j+r)$. Let $p = \lfloor\sqrt{m}\rfloor$, Let $V = \left\{(a,b) \;\middle|\; 1 \le a \le p, 1 \le b \le p\right\}$. Define the set of lines:

   $$G = \left\{ g(i,j,s,r) \;\middle|\; 1 \le j \le \lfloor p/2 \rfloor, 1 \le s \le f(n,m), 1 \le i \le s, 1 \le r < s, gcd(r,s) = 1 \right\},$$

   where $f(n,m) = c_0(n/p)^{1/3}$. Argue that the number of incidences between the lines of G and the points of V realizes the required lower bound.

1.3 Covering points by disks

(25 points)

Given a set P of n points in the plane, and a parameter k a subset $C \subseteq P$, so that $|C| = k$, is an k-cover of P, if for any point $x \in P$ there exists a point $y \in C$, so that $d(x, y) \leq r$. C_{opt} is the optimal k-cover having the minimum radius. The problem of computing C_{opt} is known as the k-center problem, and is known to be NP-Complete. However, the following approximation algorithm works quite well:

- Initialize a set $C_1 = \{p\}$ where p is any point of P.
- In the i-th stage pick the point q of P, such that the minimum distance from this point to any point of C_{i-1} is maximized (this is the furthest point from C_{i-1} in P). Let $C_i = C_{i-1} \cup \{q\}$.
- Repeat this process till C_k is computed, and output C_k.

Prove that for the covering distance, we have $dist(C_k, P) \leq 2dist(C_{opt}, P)$, where the covering distance $dist(C_k, P) = \max_{q \in P} \min_{p \in C_k} d(p, q)$. Namely, C_k is a factor 2 approximation to C_{opt}.

1.4 Covering points by a single disk

(15 points)

Give a set P of n points in $\mathbb{R}^d$, show that one can compute the minimum radius disk that contains P in linear time using linear programming in $d + 2$ dimensions.

1.5 Largest enclosed disk

(15 points)

Given a convex polytope P in $\mathbb{R}^d$ represented as the intersection of n half-spaces, show how one can compute the largest enclosed ball in P in $O(n)$ time. (Hint: use linear programming).