Computational Geometry
Exercise 3

Due: 5/1/2000

1 Exercise 3

1.1 ε-nets and VC-dimension

1. (20) Given a set of hyeprplanes L in $\mathbb{R}^d$, and two points p, q in $\mathbb{R}^d$. Show how can decide quickly detect the fact that the segment pq intersects at least half of the lines of L? How fast is the running time? What is the probability of correct answer for all such queries?

   In addition, can you ammend the above solution, so that if pq intersects less than a quarter of the lines of L it will not be reported. Analyze this variant. [hint: use ε-sample]

2. (20) Let U denote the set of all planar curves defined by $\cup_{x \in [-\infty, \infty]} (f(x), g(x))$, where $f(x), g(x)$ are polygonms of degree at most 579. Let R be the set of all regions in the plane having a curve of U as its boudnary. Prove, that the range space (X, R) has finite VC-dimension, where X is the set of points in the plane. Give a concrete upper and lower bounds on the VC-dimension of this space.

   [hint: Argue that each pair of such curves intersects at most a constant number of times. Argue that thus the number of possible assignments by such regions is polynomial by looking on the geometric interpetation of such a nassignment]

3. (20) Given a set of points P in $\mathbb{R}^d$, for $d > 2$, prove that one can construct a data-structre of size $O(n^{\lceil d/2 \rceil + \delta})$ (where δ is arbitrary), so that given a query point $p \in \mathbb{R}^d$, one can compute the *exact* nearest neighbor of p in P in $O(\log n)$ time.

   Hint 1: Use linearization to reduce the problem to verical ray-shooting on the lower nevlope of planes in $\mathbb{R}^{d+1}$. Now, use random sampling (i.e., *eps*-nets) and recursive construction to construct a tree over those planes to answer such queries. Argue that the depth of this tree is logarithmic, and that the query time (despite its large cosntant fan-out) is logarithmic. Analyze the running time.

   Hint 2: Use the fact that the lower envelope of m hyeprplanes in $d+1$ dimensions has complexity $O(m^{\lceil d/2 \rceil})$. In particular, the region below the lower envelope of thos hyeprplanes can be decomoposed into such a number of vertical prisms.

1.2 Cuttings

1. (20) Let P be a set of n points and L a set of n hyperplanes in $\mathbb{R}^d$. Prove that the minimum spanning tree of P under the intersection metric induced by L have weight $O(n^{1+(d-1)/d})$. Describe an efficient algorithm for the computation of such a tree in arbitrary dimensions. What is the running time for the planar case?

   [Hint: well, use cuttings and argue what is the right r to choose]

2. (20) Describe how to improve the running time to be near linear, by increasing the weight of the tree by a polylogarithmic factor, and allowing the algorithm to be "mistaken" (i.e., reporting a tree of weight larger than it should be). Analyze the algorithm running time, and probability for a mistake. How can this probability be decreased?

   [Hint: Use ε-nets instead of cuttings]