

# Constraint Classification: A New Approach to Multiclass Classification and Ranking

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## Abstract

We introduce constraint classification, a framework capturing many flavors of multiclass classification including multilabel classification and ranking, and present a meta-algorithm for learning in this framework. We provide generalization bounds when using a collection of  $k$  linear functions to represent each hypothesis. We also present empirical and theoretical evidence that constraint classification is more powerful than existing methods of multiclass classification.

## 1 Introduction

Multiclass classification is a central problem in machine learning, as applications that require a discrimination among several classes are ubiquitous. Examples studied in machine learning include handwritten character recognition [LS97, LBD<sup>+</sup>89], part-of-speech tagging [Bri94, RZ98], speech recognition [Jel98] and text page classification [ADW94, DKR97].

While binary classification is well understood, relatively little is known about multiclass classification. Indeed, the most common approach to multiclass classification, the “one versus all” (OvA) approach, makes direct use of “standard” binary classifiers to encode and train the output labels. The OvA scheme assumes that for each class there exists a single (simple) separator between that class and all the other classes. Another common approach, “All versus all” (AvA) [HT98], is a more expressive alternative which assumes the existence of a separator between any two classes.

“One versus all” classifiers are usually implemented using a winner-take-all (WTA) strategy that associates a real-valued function with each class in order to determine class membership. Specifically, an example belongs to the class which assigns it the highest value (i.e., the “winner”)

among all classes. While it is known that WTA is an expressive classifier [Maa00], it has limited expressivity when trained using the OvA assumption since OvA assumes that each class can be easily separated from the rest. In addition, little is known about the generalization properties or convergence of the algorithms used.

This work is motivated by several successful practical approaches, such as multiclass support vector machines (SVMs) and the sparse network of winnows (SNoW) architecture that rely on the WTA strategy over linear functions. Our aim is to improve the understanding of such classifier systems and to develop more theoretically justifiable algorithms that realize the full potential of WTA.

An alternative interpretation of WTA is that every example provides an ordering of the classes (sorted in descending order by the assigned values), where the “winner” is the first class in this ordering. It is thus natural to specify the ordering of the classes for an example *directly*, instead of implicitly through WTA.

In Section 2, we introduce *constraint classification*, where each example is labeled with a set of constraints relating multiple classes. Each such constraint specifies the relative order of two classes for this example. The goal is to learn a classifier consistent with these constraints. Learning is made possible by a simple transformation mapping each example into a set of examples (one for each constraint) and the application of any *binary* classifier on the mapped examples. In Section 3, we present a new algorithm for constraint classification that takes on the properties of the binary classification algorithm used. Therefore, using the Perceptron algorithm, it is able to learn a consistent classifier if one exists, using the winnow algorithm it can learn attribute efficiently, and using the SVM, it provides a simple implementation of multiclass SVM. The algorithm can be implemented with a subtle change to the standard (via OvA) approach to training a network of linear threshold gates. In Section 4, we present both VC-dimension and margin-based generalization bounds. Our generalization bounds apply to WTA classifiers over linear functions, for which VC-style bounds were not known. We also provide a simple analysis to derive margin-based bounds for multiclass SVMs.

In addition to multiclass classification, constraint classification generalizes multilabel classification, ranking, and of course, binary classification. As a result, our algorithm provides new insight into these problems, as well as new, powerful tools for solving them. For example, in Section 3.3, we show that the OvA assumption can cause learning to fail, even when a consistent classifier exists. Section 5 provides empirical evidence that the constraint classification outperforms the OvA approach.

## 2 Constraint Classification

Learning problems often assume that examples,  $(x, y) \in \mathcal{X} \times \mathcal{Y}$ , are drawn *i.i.d.* from fixed probability distribution,  $\mathcal{D}_{\mathcal{X} \times \mathcal{Y}}$ , over  $\mathcal{X} \times \mathcal{Y}$ .  $\mathcal{X}$  is referred to as the instance space and  $\mathcal{Y}$  is referred to as the output space (*label set*).

**Definition 2.1 (Learning)** Given  $m$  examples,  $S = ((x_1, y_1), \dots, (x_m, y_m))$ , drawn *i.i.d.* from  $\mathcal{D}_{\mathcal{X} \times \mathcal{Y}}$ , a hypothesis class  $\mathcal{H}$  and an error function  $\mathcal{E} : \mathcal{X} \times \mathcal{Y} \times \mathcal{H} \rightarrow \{0, 1\}$ , a learning algorithm

$\mathcal{L}(S, \mathcal{H})$  attempts to output a function  $h \in \mathcal{H}$ , where  $h : \mathcal{X} \rightarrow \mathcal{Y}$ , that minimizes the expected error on a randomly drawn example.

**Definition 2.2 (Permutations)** Denote the set of full orders over  $\{1, \dots, k\}$  as  $S^k$ , consisting of all permutations of  $\{1, \dots, k\}$ . Similarly,  $\bar{S}^k$  denotes the set of all partial orders over  $\{1, \dots, k\}$ . A partial order,  $c \in \bar{S}^k$ , defines a binary relation,  $\prec_c$  and can be represented by set of pairs on which  $\prec_c$  holds,  $c = \{(i, j) | i \prec_c j\}$ . In addition, for any set of pairs  $c = \{(i_1, j_1), \dots, (i_n, j_n)\}$ , we refer to  $c$  both as a set of pairs and as the partial order produced by the transitive closure of  $c$  with respect to  $\prec_c$ . Given two partial orders  $a, b \in \bar{S}^k$ ,  $a$  is *consistent* with  $b$  (denoted  $a \sqsubseteq b$ ) if for every  $(i, j) \in \{1, \dots, k\}^2$ ,  $i \prec_b j$  holds whenever  $i \prec_a j$ . If  $c \in S^k$  is a full order, then it can be represented by a list of  $k$  integers where  $i \prec_c j$  if  $i$  precedes  $j$  in the list. The size of a partial order,  $|c|$  is the number of pairs specified in  $c$ .

**Definition 2.3 (Constraint Classification)** Constraint classification is a learning problem where each example  $(x, y) \in \mathcal{X} \times \bar{S}^k$  is labeled according to a partial order  $y \in \bar{S}^k$ . A constraint classifier,  $h : \mathcal{X} \rightarrow \bar{S}^k$ , is consistent with example  $(x, y)$  if  $y$  is consistent with  $h(x)$  ( $y \sqsubseteq h(x)$ ). When  $|y| \leq c$ , we call it *c-constraint classification*.

**Definition 2.4 (Error Indicator Function)** For any  $(x, y) \in \mathcal{X} \times \bar{S}^k$ , and hypothesis  $h : \mathcal{X} \rightarrow \bar{S}^k$ , the *indicator function*  $\mathcal{E}(x, y, h)$  indicates an error on example  $x$ ,  $\mathcal{E}(x, y, h) = 1$  if  $y \not\sqsubseteq h(x)$ , and 0 otherwise.

For example, if  $k = 4$  and example  $(x, y) = (x, \{(2, 3), (2, 4)\})$ ,  $h_1(x) = (2, 3, 1, 4)$ , and  $h_2(x) = (4, 2, 3, 1)$ , then  $h_1$  is correct since 2 precedes 3 and 2 precedes 4 in the full order  $(2, 3, 1, 4)$  whereas  $h_2$  is incorrect since 4 precedes 2 in  $(4, 2, 3, 1)$ .

**Definition 2.5 (Error)** Given an example  $(x, y)$  drawn *i.i.d.* from  $\mathcal{D}_{\mathcal{X} \times \mathcal{Y}}$ , the *true error* of  $h \in \mathcal{H}$ , where  $h : \mathcal{X} \rightarrow \mathcal{Y}'$  is defined to be  $err(h) = \Pr_{\mathcal{D}}[\mathcal{E}(x, y, h)]$ . Given  $S = ((x_1, y_1), \dots, (x_m, y_m))$ , the *empirical error* of  $h \in \mathcal{H}$  with respect to  $S$  is defined to be  $err(S, h) = \frac{1}{|S|} \sum_{(x, y) \in S} \mathcal{E}(x, y, h)$ .

In this paper, we consider constraint classification problems where hypotheses are functions from  $\mathbb{R}^d$  to  $S^k$  that output a permutation of  $\{1, \dots, k\}$ .

**Definition 2.6 (Linear Sorting Function)** Let  $w = (w_1, \dots, w_k)$  be a set of  $k$  vectors, where  $(w_1, \dots, w_k) \in \mathbb{R}^d$ . Given  $x \in \mathbb{R}^d$ , a *linear sorting classifier* is a function  $h : \mathbb{R}^d \rightarrow S^k$  computed in the following way:

$$h(x) = \underset{i=1 \dots k}{\text{argsort}} w_i \cdot x,$$

where  $\text{argsort}$  returns a permutation of  $\{1, \dots, k\}$  where  $i$  precedes  $j$  if  $w_i \cdot x > w_j \cdot x$ . In the case that  $w_i \cdot x = w_j \cdot x$ ,  $i$  precedes  $j$  if  $i < j$ .

Constraint classification can model many well-studied learning problems including multiclass classification, ranking and multilabel classification. Table 1 shows a few interesting classification problems expressible as constraint classification. It is easy to show:

Problem	Internal Representation	Output Space ( $\mathcal{Y}$ )	Hypothesis	Size of Mapping
binary	$w \in \mathbb{R}^d$	$\{-1, 1\}$	$\text{sign } w \cdot x$	1
multiclass	$(w_1, \dots, w_k) \in \mathbb{R}^{kd}$	$\{1, \dots, k\}$	$\text{argmax}_{\{1, \dots, k\}} w_i \cdot x$	$k - 1$
$l$ -multilabel	$(w_1, \dots, w_k) \in \mathbb{R}^{kd}$	$\{1, \dots, k\}^l$	$\text{argmax}_{\{1, \dots, k\}}^l w_i \cdot x$	$l(k - 1)$
ranking	$(w_1, \dots, w_k) \in \mathbb{R}^{kd}$	$S^k$	$\text{argsort}_{\{1, \dots, k\}} w_i \cdot x$	$k - 1$
constraint*	$(w_1, \dots, w_k) \in \mathbb{R}^{kd}$	$\bar{S}^k$	$\text{argsort}_{\{1, \dots, k\}} w_i \cdot x$	–
$c$ -constraint*	$(w_1, \dots, w_k) \in \mathbb{R}^{kd}$	$\bar{S}_c^k$	$\text{argsort}_{\{1, \dots, k\}} w_i \cdot x$	$c$

Table 1: Definitions for various learning problems (notice that the hypothesis for constraint classification is always a full order) and the size of the resultant mapping to  $c$ -constraint classification.  $\text{argmax}^l$  is a variant of  $\text{argmax}$  that returns the  $l$  maximal indices with respect to  $w_i \cdot x$ .  $\text{argsort}$  is a linear sorting function (see Definition 2.6).

**Lemma 2.7 (Problem mappings)** *All of the learning problems in Table 1 can be expressed as constraint classification problems.*

Consider a 4-class multiclass example,  $(x, 3)$ . It is transformed into the 3-constraint example,  $(x, \{(3, 1), (3, 2), (3, 4)\})$ . If we find a constraint classifier that correctly labels  $x$  according to the given constraints where  $w_3 \cdot x > w_1 \cdot x$ ,  $w_3 \cdot x > w_2 \cdot x$ , and  $w_3 \cdot x > w_4 \cdot x$ , then  $3 = \text{argmax}_{1,2,3,4} w_i \cdot x$ . If instead we are given a ranking example  $(x, \{(3, 2, 1, 4)\})$ , it can be transformed into  $(x, \{(3, 2), (2, 1), (1, 4)\})$ .

## 3 Learning

In this section,  $k$ -class constraint classification is transformed into binary classification in higher dimension. Each example  $(x, y) \in \mathbb{R}^d \times \bar{S}^k$  becomes a set of examples in  $\mathbb{R}^{kd} \times \{-1, 1\}$  with each constraint  $(i, j)$  contributing a single ‘positive’ and a single ‘negative’ example. Then, a separating hyperplane for the expanded example set (in  $\mathbb{R}^{kd}$ ) can be viewed as a linear sorting function over  $k$  linear functions, each in  $d$  dimensional space.

### 3.1 Transformation

**Definition 3.1 (Chunk)** A vector  $\mathbf{v} = (v_1, \dots, v_{kd}) \in \mathbb{R}^{kd} = \mathbb{R}^d \times \dots \times \mathbb{R}^d$ , is broken into  $k$  chunks  $(\mathbf{v}_1, \dots, \mathbf{v}_k)$  where the  $i$ -th chunk,  $\mathbf{v}_i = (v_{(i-1)*d+1}, \dots, v_{i*d})$ .

**Definition 3.2 (Expansion)** Let  $\text{Vec}(x, i)$  be a vector  $x \in \mathbb{R}^d$  embedded in  $kd$  dimensions, by writing the coordinates of  $x$  in the  $i$ -th chunk of a vector in  $\mathbb{R}^{k(d+1)}$ . Denote by  $\mathbf{0}^l$  the zero vector of length  $l$ . Then  $\text{Vec}(x, i)$  can be written formally as the concatenation of three vectors,  $\text{Vec}(x, i) = (\mathbf{0}^{(i-1)*d}, x, \mathbf{0}^{(k-i)*d}) \in \mathbb{R}^{kd}$ . Finally,  $\text{Vec}(x, i, j) = \text{Vec}(x, i) - \text{Vec}(x, j)$ , is the embedding of  $x$  in the  $i$ -th chunk and  $-x$  in the  $j$ -th chunk of a vector in  $\mathbb{R}^{kd}$ .

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Algorithm CONSTRCLASSLEARN
INPUT:
     $S = ((x_1, y_1), \dots, (x_m, y_m)),$ 
    where  $S \in \{\mathbb{R}^d \times \bar{S}^k\}^m$ 
OUTPUT: A classifier  $h'$ 
begin

$$\mathcal{H} = \left\{ h \mid \begin{array}{l} h : \mathbb{R}^{kd} \rightarrow \{-1, 1\} \\ h(\mathbf{x}) = \mathbf{v} \cdot \mathbf{x}, \\ \mathbf{v}, \mathbf{x} \in \mathbb{R}^{kd} \end{array} \right\}$$

    Calculate  $\mathbf{P}(S) \in \mathbb{R}^{kd} \times \{1, -1\}$ 
     $h = \mathcal{L}(\mathbf{P}(S), \mathcal{H}) \in \mathcal{H}$ 
    Set  $h'(x) = \text{argsort}_{1, \dots, k} \mathbf{v}_i \cdot x$ 
end

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(a)

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Algorithm ONLINECONCLASSLEARN
INPUT:
     $S = ((x_1, y_1), \dots, (x_m, y_m)),$ 
    where  $S \in \{\mathbb{R}^d \times \bar{S}^k\}^m$ 
OUTPUT: A classifier  $h'$ 
begin
    Initialize  $(\mathbf{v}_1, \dots, \mathbf{v}_k) \in \mathbb{R}^{kd}$ 
    Repeat until converge
    for  $i = 1..m$  do
        for all  $(j, j') \in y_i$  do
            if  $\mathbf{v}_j \cdot x_i < \mathbf{v}_{j'} \cdot x_i$  then
                promote( $\mathbf{v}_j$ )
                demote( $\mathbf{v}_{j'}$ )
    Set  $h'(x) = \text{argsort}_{1, \dots, k} \mathbf{v}_i \cdot x$ 
end

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(b)

Figure 1: (a) Meta-learning algorithm for constraint classification with linear sorting functions (see Definition 2.6).  $\mathcal{L}(\cdot, \cdot)$  is any binary learning algorithm returning a separating hyperplane. (b) Online meta-algorithm for constraint classification with linear sorting functions (see Definition 2.6). The particular online algorithm used determines how  $(\mathbf{v}_1, \dots, \mathbf{v}_k)$  is initialized and the promotion and demotion strategies.

**Definition 3.3 (Expanded Example Sets)** Given an example  $(x, y)$ , where  $x \in \mathbb{R}^d$  and  $y \in \bar{S}^k$ , we define the *expansion* of  $(x, y)$  into a set of examples as follows,

$$\mathbf{P}_+(x, y) = \left\{ (\text{Vec}(x, i, j), 1) \mid (i, j) \in y \right\} \subseteq \mathbb{R}^{kd} \times \{1\},$$

A set of negative examples is defined as the reflection of each expanded example through the origin, specifically

$$\mathbf{P}_-(x, y) = \left\{ (-\mathbf{x}, -1) \mid (\mathbf{x}, 1) \in \mathbf{P}_+(x, y) \right\} \subseteq \mathbb{R}^{kd} \times \{-1\},$$

and the set of both positive and negative examples is denoted by  $\mathbf{P}(x, y) = \mathbf{P}_+(x, y) \cup \mathbf{P}_-(x, y)$ . The expansion of a set of examples,  $S$ , is defined as the union of all of the expanded examples in the set,

$$\mathbf{P}(S) = \bigcup_{(x, y) \in S} \mathbf{P}(x, y) \subseteq \mathbb{R}^{kd} \times \{-1, 1\}.$$

## 3.2 Algorithm

We present a meta-learning algorithm for constraint classification that finds a linear sorting function by using any algorithm that attempts to return a separating hyperplane for a set of binary

labeled examples (see Figure 1 (a)). Given a set of examples  $S \subseteq \mathbb{R}^d \times \bar{S}^k$ , the algorithm simply finds a separating hyperplane  $h(\mathbf{x}) = \mathbf{v} \cdot \mathbf{x}$  for  $\mathbf{P}(S) \subseteq \mathbb{R}^d \times \{-1, 1\}$ . Suppose  $h$  correctly classifies  $(\mathbf{x}, 1) = (\text{Vec}(x, i, j), 1) \in \mathbf{P}(S)$ , then  $\mathbf{x} \cdot \mathbf{v} = x \cdot \mathbf{v}_i - x \cdot \mathbf{v}_j > 0$ , and the constraint  $(i, j)$  on  $x$  (dictating that  $x \cdot \mathbf{v}_i > x \cdot \mathbf{v}_j$ ) is consistent with  $h(\mathbf{x})$ . Therefore, if  $h(\mathbf{x})$  correctly classifies all  $x \in \mathbf{P}(S)$ , then  $\text{argsort}_{1, \dots, k} \mathbf{v}_i \cdot x$  is a consistent linear sorting function.

This framework is significant to multiclass classification in many ways. First, the hypothesis learned above is more expressive than when the OvA assumption is used. Second, it is easy to verify that other algorithmic-specific properties are maintained by the above transformation. For example, attribute efficiency is preserved when using the winnow algorithm. Finally, the multiclass support vector machine can be implemented by learning a maximal margin (see Definition 4.3) hyperplane to separate  $\mathbf{P}(S)$ .

### 3.3 Comparison to “One vs. All”

A common approach to multiclass classification ( $\mathcal{Y} = \{1, \dots, k\}$ ) is to make the *one-versus-all* (OvA) assumption, namely, that each class can be separated from the rest using a binary classification algorithm. Learning proceeds by learning  $k$  independent binary classifiers, one corresponding to each class, where example  $(x, y)$  is considered positive for classifier  $y$  and negative for all others.

It is easy to construct an example where the OvA assumption causes the learning to fail even when there exists a consistent linear sorting function. (see Figure 2) Notice, since the existence of a consistent linear sorting function (w.r.t.  $S$ ) implies the existence of a separating hyperplane (w.r.t.  $\mathbf{P}(S)$ ), any learning algorithm guaranteed to separate two separable point sets (e.g. the Perceptron algorithm) is guaranteed to find a consistent linear sorting function. In Section 5, we use the perceptron algorithm to find a consistent classifier for an extension of the example in Figure 2 to  $\mathbb{R}^{100}$  when OvA fails.

### 3.4 Comparison to Networks of Linear Threshold Gates (Perceptron)

It is possible to implement the algorithm in Section 3.2 using a network of linear classifiers such as multi-output Perceptron [AB99], SNoW [CCRR99, Rot98], and multiclass SVM [CS00, WW99]. Such a network has  $x \in \mathbb{R}^d$  as input and  $k$  outputs, each represented by a weight vector,  $w_i \in \mathbb{R}^d$ , where the  $i$ -th output computes  $w_i \cdot x$  (see Figure 1 (b)).

Typically, a label is mapped, via fixed transformation, into a  $k$ -dimensional output vector, and each output is trained separately, as in the OvA case. Alternately, if the online perceptron algorithm is plugged into the meta-algorithm in Section 3.2, then updates are performed according to a dynamic transformation. Specifically, given  $(x, y)$ , for every constraint  $(i, j) \in y$ , if  $w_i \cdot x < w_j \cdot x$ ,  $w_i$  is ‘promoted’ and  $w_j$  is ‘demoted’. Using a network in this results in an ultraconservative online algorithm for multiclass classification [CS01]. This subtle change enables the commonly used network of linear threshold gates to learn every hypothesis it is capable of representing.

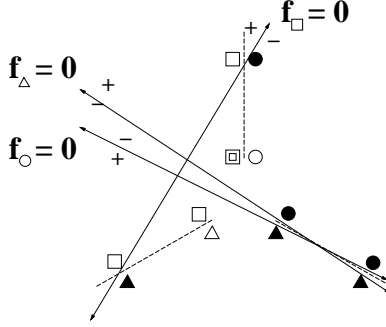


Figure 2: A 3-class classification example in  $\mathbb{R}^2$  showing that one-versus-all (OvA) does not converge to a consistent hypothesis. Three classes (squares, triangles, and circles) should be separated from the rest. Solid points act as 10 points in their respective classes. The OvA assumption will attempt to separate the circles from squares and triangles with a single separating hyperplane, as well as the other 2 combinations. Because the solid points are weighted, all OvA classifiers are required to classify them correctly or suffer 10 mistakes, thus restricting what the final hypotheses will be. As a result, the OvA assumption will misclassify point outlined with a double square since the square classifier predicts “not square” and the circle classifier predicts “circle”. One can verify that there exists a WTA classifier for this example..

## 4 Generalization Bounds

A PAC-style analysis of multiclass functions that uses an extended notion of VC-dimension for multiclass case [BCHL95] provides poor bounds on generalization for WTA, and the current best bounds rely on a generalized notion of margin [ASS00]. In this section, we prove tighter bounds using the new framework.

We seek generalization bounds for learning with  $\mathcal{H}$ , the class of linear sorting functions (Definition 2.6). Although both VC-dimension-based (based on growth function) and margin-based bounds for the class of hyperplanes in  $\mathbb{R}^{kd}$  are known [Vap98, AB99], they cannot directly be applied since  $\mathbf{P}(S)$  produces points that are random, but not *independently* drawn. It turns out that bounds can be derived indirectly by using known bounds for constraint classification. Due to space considerations, all proofs have been omitted.

**Theorem 4.1 (VC-Style Bound)** *For any  $0 < \delta < 1$ , any  $h \in \mathcal{H}$ , the class of linear sorting functions over  $k$  linear functions in  $\mathbb{R}^d$ , where  $h : \mathbb{R}^d \rightarrow S^k$ , given  $S = ((x_1, y_1), \dots, (x_m, y_m))$ , a sample of size  $m$  drawn *i.i.d.* from  $\mathcal{D}_{\mathbb{R}^d \times S^k}$ , with probability at least  $1 - \delta$ ,*

$$err(h) \leq err(S, h) + \sqrt{4 \frac{kd \log(2emk/d) - \log \delta/4}{m}} + \frac{1}{m}$$

Theorem 4.1 can also be improved in the standard way [Vap98] for the case where there is no training error.

**Corollary 4.2** *To guarantee that the sampled error differ from the true error by less than  $\varepsilon$  with probability at least  $1 - \delta$ , it is sufficient to draw  $m > m(\varepsilon, \delta)$  examples drawn i.i.d. from  $\mathcal{D}_{\mathbb{R}^d \times \bar{S}^k}$  where,  $m(\varepsilon, \delta) = O\left(\frac{1}{\varepsilon^2} \max\left(kd \log \frac{kd}{\varepsilon^2}, \log \frac{1}{\delta}\right)\right)$ . If we are able to find a classifier  $h$  which is consistent on all the examples (i.e.  $\text{err}(S, h) = 0$ ), then to achieve true error less than  $\varepsilon$ , we need to pick  $m > m_1(\varepsilon, \delta)$ , where  $m_1(\varepsilon, \delta) = O\left(\frac{1}{\varepsilon} \max\left(kd \log \frac{kd}{\varepsilon^2}, \log \frac{1}{\delta}\right)\right)$ .*

In light of the transformation described in Section 3.1, a natural extension of binary margin to the constraint case can be derived. The constraint margin of a set of a set of examples labeled with constraints,  $S$ , is simply the binary margin of  $\mathbf{P}(S)$ . It turns out that constraint margin is equivalent to the multiclass margin defined ad-hoc in [CS00, WW99].

**Definition 4.3 (Constraint Margin)** The margin of an example  $(x_i, y_i)$ , where  $(x_i, y_i) \in \mathbb{R}^d \times \bar{S}^k$ , with respect to a linear sorting function,  $h(x) = \text{argsort}_{\{1, \dots, k\}} w_i \cdot x$ , where  $w_i, x \in \mathbb{R}^d$ , is

$$\gamma_i = \min_{(j, j') \in y_i} (w_j \cdot x_i - w_{j'} \cdot x_i).$$

The minimal margin over a set of examples,  $S = ((x_1, y_1), \dots, (x_m, y_m))$ , is

$$\text{mar}(h) = \min_S \gamma_i.$$

**Theorem 4.4 (Constraint Margin Bound)** *Consider real-valued linear sorting functions  $\mathcal{H}$  with  $\sum_{i=1, \dots, k} \|w_i\| = 1$ , where  $h : \mathbb{R}^d \rightarrow S^k$ , and fix  $\gamma \in \mathbb{R}^+$ . For any probability distribution  $\mathcal{D}$  on  $\mathbb{R}^d \times \bar{S}_C^k$  with support in a ball of radius  $R$  around the origin, with probability  $1 - \delta$  over  $m$  random examples  $S$ , any hypothesis  $h \in \mathcal{H}$  that has constraint margin  $\text{mar}_S(h) \geq \gamma$  on  $S$  has error no more than*

$$\text{err}_{\mathcal{D}}(h) = \epsilon(\gamma, m, \delta, R, C) \leq \frac{2C}{m} \left( \frac{256R^2}{\gamma^2} \log \frac{em\gamma}{32R^2} \log \frac{32m}{\gamma^2} + \log \frac{4}{\delta} \right)$$

provided  $m > \frac{2}{\epsilon}$  and  $\frac{256R^2}{\gamma^2} < m$ .

## 5 Experiments

As in previous multiclass classification work [DB95, ASS00], we tested our algorithm on a suite of problems from the Irvine Repository of machine learning [BM98] (see Table 2). In addition, we created a simple experiment using synthetic data. The data was generated according to a WTA function over 3 randomly generated linear functions in  $\mathbb{R}^{100}$ , each with weight vectors inside the unit ball. Then, 50K training and 50K testing examples were randomly sampled within a ball of radius 2 around the origin and labeled with the linear function that produced the highest value.

A comparison is made between the OvA approach (Section 3.3) and the constraint classification approach. Both were implemented on the same network of multi-output Perceptron network with  $k(d+1)$  weights (with one threshold per class). Constraint classification used the modified update

Dataset	Features	Classes	Training Examples	Testing Examples
glass	9	6	214	–
vowel	10	11	528	462
soybean	35	19	307	376
audiology	69	24	200	26
ISOLET	617	26	6238	1559
letter	16	26	16000	4000
Synthetic*	100	3	50000	50000

Table 2: Summary of problems from the UCI repository. The synthetic data is sampled from a random linear sorting function (see Section 5).

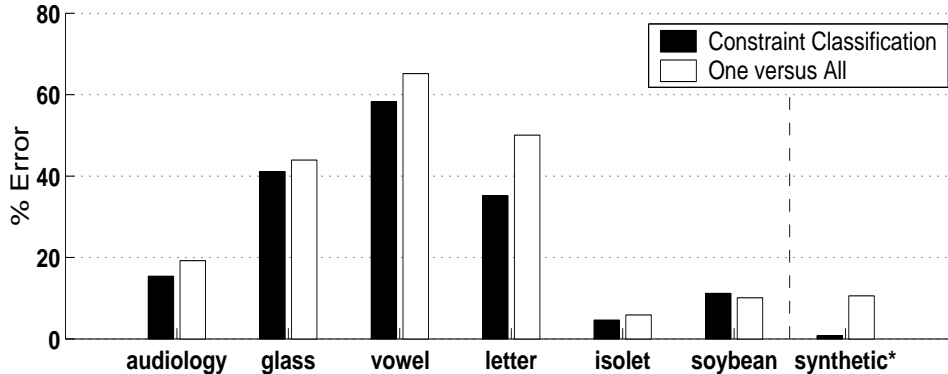


Figure 3: Comparison of constraint classification meta-algorithm using the Perceptron algorithm to multi-output Perceptron using the OvA assumption. All of the results for the constraint classification algorithm are competitive with the known. The synthetic data would converge to 0 error using constraint classification but would not converge using the OvA approach.

rule discussed in Section 3.4. Each update was performed as follows:  $w_{t+1} = w_t + x$  for promotion and  $w_{t+2} = w_t - x$  for demotion. The networks were initialized with weights all 0.

For each multiclass example  $(x, y_m) \in \mathbb{R}^d \times \{1, \dots, k\}$ , a constraint classification example  $(x, y_c) \in \mathbb{R}^d \times \bar{S}^k$  was created, where  $y_c = \{(y_m, i) \mid j = \{1, \dots, k\} \setminus y_m\}$ . Notice error (Definition 2.4) of  $(x, y_c)$  corresponds to the traditional error for multiclass classification.

Figure 3 shows that constraint classification outperforms the multioutput Perceptron when using the OvA assumption.

## 6 Conclusions

The view of multiclass classification presented here simplifies the implementation, analysis, and understanding of many preexisting approaches. Multiclass support vector machines, ultraconservative online algorithms, and traditional one-versus-all approaches can be cast in this framework.

It would be interesting to see how this method could be combined with the error-correcting output coding method in [DB95] that provides another way to extend the OvA approach. Furthermore, this view allows for a very natural extension of multiclass classification to constraint classification – capturing within it complex learning tasks such as multilabel classification and ranking. Because constraint classification is a very intuitive approach and its implementation can be carried out by any discriminant technique, and not only by optimization techniques, we think it will have useful real-world applications.

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