On the Fermat-Weber Center of a Convex Object

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Abstract

We show that for any convex object Q in the plane, the average distance from the Fermat-Weber center of Q to the points in Q is at least $\Delta(P)/7$, where $\Delta(P)$ is the diameter of P, and that there exists a convex object for which this distance is $\Delta(P)/6$. We use this result to obtain a linear-time approximation scheme for finding an approximate Fermat-Weber center of a convex polygon Q.

1 Introduction

For a planar object Q and a point y, let $\mu_Q(y)$ be the average distance between y and the points in Q, that is, $\mu_Q(y) = \int_{x \in Q} \|xy\| \, dx / \text{area}(Q)$, where $\|xy\|$ is the Euclidean distance between x and y. Let $\mathcal{FW}_Q$ be a point for which this average distance is minimal, that is, $\mu_Q^* = \mu_Q(\mathcal{FW}_Q) = \min_{y \in Q} \mu_Q(y)$. $\mathcal{FW}_Q$ is a *Fermat-Weber* center of Q.

In this paper we restrict our attention to convex objects. It is easy to see that for such objects Q, $\mathcal{FW}_Q \in Q$. The paper is composed of two parts. In the first part we study the relation between μ_Q^* and the diameter of Q, denoted $\Delta(Q)$. In the second part of the paper we present an efficient algorithm that finds a point $p \in Q$ that is a good approximation of $\mathcal{FW}_Q$, in the sense that $\mu_Q(p)$ is not much greater than μ_Q^*.

For a disk D, it is easy to verify that the Fermat-Weber center of D coincides with the center of D and that $\mu_D^* = \Delta(D)/3$. This raises the question: Does there exist a constant c, such that for *any* convex object Q, $\mu_Q^* \geq c\Delta(Q)$, and, if yes, what is the largest such constant c^*. In Section 2 we show that the answer is indeed yes, and that $1/7 \leq c^* \leq 1/6$. We use this result in Section 3 to obtain an approximation algorithm that, given a convex n-gon Q and a parameter $\varepsilon > 0$, finds in linear time a point $p \in Q$, such that $\mu_Q(p) \leq (1+\varepsilon)\mu_Q^*$.

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The Fermat-Weber center of an object Q is of course a very significant point of Q. It is, e.g., the ideal location for a fire station serving the region Q. The classical Fermat-Weber problem is to find a point in a set F of feasible facility locations, that minimizes the average distance to the points in a set D of (possibly weighted) demand locations. If D is a finite set of points, F is the entire plane, and distances are measured using the L_2 metric, then it is known that the solution is algebraic (see [1]). Chandrasekaran and Tamir [3] and Bose, Maheshwari and Morin [2] give polynomial-time approximation schemes based on the ellipsoid method and on data structures for answering average distance queries, respectively. Under the L_1 metric an exact solution can be computed in linear time. See Wesolowsky [5] for a survey of the Fermat-Weber problem.

Only a few papers deal with the continuous version of the Fermat-Weber problem, where the set of demand locations is continuous. The most recent by Fekete, Mitchell and Weinbrecht [4] who present algorithms for computing an optimal solution when $D = F = P$ is a simple polygon or a polygon with holes, and the distance between two points in P is the L_1 geodesic distance between them. They also consider several related problems and include references to more previous work. This paper also deals with the continuous version of the Fermat-Weber problem. It provides a linear-time approximation scheme for the case where P is a convex polygon.

2 $1/7 \leq c^* \leq 1/6$

We first show that $c^* \geq 1/7$.

Theorem 2.1 *Let P be a convex object. Then $\mu^*_P \geq \Delta(P)/7$.*

Proof: Let $\mathcal{FW}_P$ be a Fermat-Weber center of P. We need to show that $\int_{x \in P} \|x\mathcal{FW}_P\| \, dx \geq \frac{\Delta(P)}{7}\text{area}(P)$. We do this in two stages. In the first stage we show that for a certain subset P' of P of area area$(P)/2$ the sum of distances between $\mathcal{FW}_P$ and the points in P' is relatively large. More precisely, we show that $\int_{x \in P'} \|x\mathcal{FW}_P\| \, dx \geq \frac{\Delta(P)}{8}\text{area}(P)$. This implies that for any convex object Q, $\mu^*_Q \geq \Delta(Q)/8$. In the second stage we apply this intermediate result to a collection of convex subsets of $P - P'$ that are pair-wise disjoint to obtain the claimed result.

We now describe the first stage. Let s be a line segment of length $\Delta(P)$ connecting two points p and q on the boundary of P. We may assume that s is horizontal and that p is its right endpoint, since we can always rotate P around, say, p until this is the case.

Draw a vertical line l_0 through the center point of s. We divide the part of P to the right of l_0 into $n+1$ slabs by drawing n evenly-spaced vertical lines $l_1, \ldots, l_n$; see Figure 1. Similarly, we divide the part of P to the left of l_0 into $n+1$ slabs by drawing the vertical lines $l'_1, \ldots, l'_n$.

Let P^α be the polygon obtained from P by shrinking it by a factor of α, that is, by applying the transformation $f(a,b) = (a/\alpha, b/\alpha)$ to the points (a,b) in P. We place a copy Q_1 of P^2, such that, Q_1 is contained in P and has a common tangent with P at the endpoint p. Similarly, we place a copy Q'_1 of P^2, such that, Q'_1 is contained in P and has a common tangent with P at q. Clearly, $Q_1 \cap Q_2 = \emptyset$ and $\text{area}(Q_1) = \text{area}(Q'_1) = \frac{\text{area}(P)}{4}$.

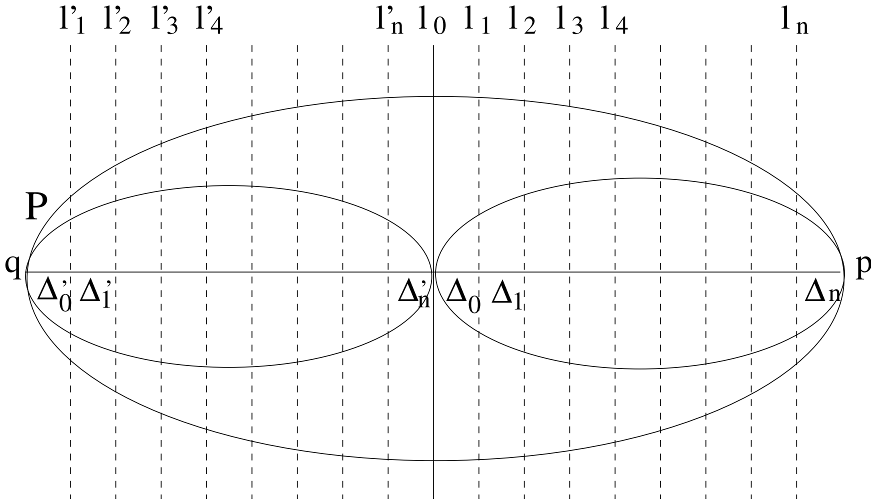

Figure 1: Proof of intermediate result.

The lines $l_1, \ldots, l_n$ divide Q_1 into $n+1$ parts denoted $\Delta_0, \ldots, \Delta_n$, and the lines $l'_1, \ldots, l'_n$ divide Q_2 into $n+1$ parts denoted $\Delta'_0, \ldots, \Delta'_n$, see Figure 1, such that, the parts Δ_i and Δ'_i are congruent, for $i = 0..n$. In particular, $\text{area}(\Delta_i) = \text{area}(\Delta'_i)$, for $i = 0..n$.

The point $\mathcal{FW}_P$ lies either to the left of l'_1, or between two consecutive vertical lines, or to the right of l_n. However, for any $0 \leq i \leq n$, regardless of the exact location of $\mathcal{FW}_P$, we have that

$$\int_{x \in \Delta_i} \|x\mathcal{FW}_P\|\, dx + \int_{x \in \Delta'_i} \|x\mathcal{FW}_P\|\, dx > (n-1)\frac{\Delta(P)}{2(n+1)}\text{area}(\Delta_i)\ ,$$

so by summing over i we get that

$$\int_{x \in Q_1} \|x\mathcal{FW}_P\|\, dx + \int_{x \in Q_2} \|x\mathcal{FW}_P\|\, dx > (n-1)\frac{\Delta(P)}{2(n+1)}\text{area}(Q_1) = (n-1)\frac{\Delta(P)}{8(n+1)}\text{area}(P)\ ,$$

and by letting n tend to infinity we obtain our intermediate result, namely that

$$\int_{x \in Q_1} \|x\mathcal{FW}_P\|\, dx + \int_{x \in Q_2} \|x\mathcal{FW}_P\|\, dx \geq \frac{\Delta(P)}{8}\text{area}(P)\ .$$

This intermediate result immediately implies that for any convex object Q, $\mu^*_Q \geq \Delta(Q)/8$. In the remaining part of the proof we show that the 8 in the denominator can be replaced by 7.

Consider Figure 2. We draw the axis-aligned bounding box of P. The line segment s (whose length is $\Delta(P)$) divides the bounding box of P into two rectangles — $abpq$ above s and $qpcd$ below s. We divide each of these rectangles into two parts (a lower part and an upper part) of equal area, by drawing the two horizontal lines l and l'. Let Q_2 denote the intersection of P with the upper part of the upper rectangle, and let Q'_2 denote the intersection of P with the lower part of the lower rectangle.

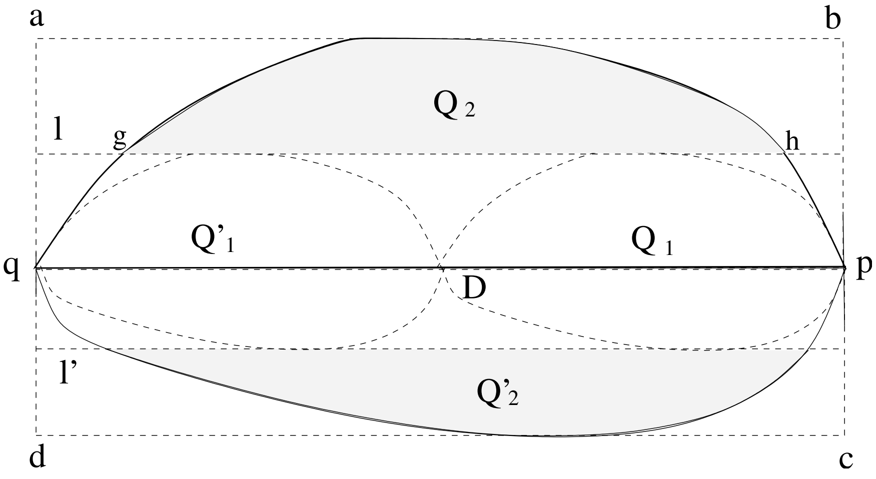

Figure 2: Proof of improved result.

Let us verify some facts concerning Q_2 and Q_2'. $Q_2 \cap Q_2' = \emptyset$, $Q_2 \cap Q_1 = \emptyset$, $Q_2 \cap Q_1' = \emptyset$, $Q_2' \cap Q_1 = \emptyset$, and $Q_2' \cap Q_1' = \emptyset$. Notice also that $\Delta(Q_2), \Delta(Q_2') \geq \Delta(P)/2$, since the length of the line segment $l \cap Q_2$ (alternatively, the line segment $l' \cap Q_2'$) is at least $\Delta(P)/2$.

We next observe that $\text{area}(Q_2) + \text{area}(Q_2') \geq \text{area}(P)/4$ by showing that $\text{area}(Q_2) \geq \text{area}(P \cap abpq)/4$ (and that $\text{area}(Q_2') \geq \text{area}(P \cap qpcd)/4$). Let g, h be the two points on the line l that also lie on the boundary of Q_2, and let e be any point on the segment ab that also lies on the boundary of Q_2. Let $l(s)$ be the line containing s, and let T be the triangle defined by $l(s)$ and the two line segments connecting e to $l(s)$ and passing through g and through h, respectively. Let T_2 denote the triangle geh.

Clearly $T_2 \subseteq Q_2$. Put $R = Q_2 - T_2$. Then $\text{area}(Q_2) = \text{area}(T_2) + \text{area}(R) = \text{area}(T)/4 + \text{area}(R)$. Therefore $\text{area}(Q_2) \geq (\text{area}(T) + \text{area}(R))/4 \geq \text{area}(P \cap abpq)/4$. We show that $\text{area}(Q_2') \geq \text{area}(P \cap qpcd)/4$ using the "symmetric" construction. Since $(P \cap abpq) \cup (P \cap qpcd) = P$ we obtain that $\text{area}(Q_2) + \text{area}(Q_2') \geq \text{area}(P)/4$.

Now using the implication of our intermediate result we have

$$\begin{aligned}
\int_{x \in Q_2} \|x\mathcal{FW}_{Q_2}\| \, dx + \int_{x \in Q_2'} \|x\mathcal{FW}_{Q_2'}\| \, dx &\geq \frac{\Delta(Q_2)}{8}\text{area}(Q_2) + \frac{\Delta(Q_2')}{8}\text{area}(Q_2') \\
&\geq \frac{\Delta(P)}{16}(\text{area}(Q_2) + \text{area}(Q_2')) \geq \frac{\Delta(P)}{64}\text{area}(P) \; .
\end{aligned}$$

Therefore

$$\begin{aligned}
\int_{x \in P} \|x\mathcal{FW}_P\| \, dx &\geq \int_{x \in Q_1} \|x\mathcal{FW}_P\| \, dx + \int_{x \in Q_1'} \|x\mathcal{FW}_P\| \, dx + \int_{x \in Q_2} \|x\mathcal{FW}_P\| \, dx \\
&\quad + \int_{x \in Q_2'} \|x\mathcal{FW}_P\| \, dx \geq \frac{\Delta(P)}{8}\text{area}(P) + \frac{\Delta(P)}{64}\text{area}(P) = \frac{9\Delta(P)}{64}\text{area}(P) \; .
\end{aligned}$$

At this point we may conclude that for any convex object Q, $\mu_Q^* \geq 9\Delta(Q)/64$. So we repeat the calculation above using this result for the regions Q_2 and Q_2' (instead of using the slightly weaker result, i.e., $\mu_Q^* \geq \Delta(Q)/8$). This calculation will yield a slightly stronger result, etc. etc. It is easy to verify that this sequence of results converges to $\mu_Q^* \geq \Delta(Q)/7$. ■

We now show that $c^* \leq 1/6$.

Theorem 2.2 *There exists a convex object P such that $\mu_P^* \leq \Delta(P)/6$.*

Proof: Consider the rhombus P shown in Figure 3. It is easy to verify that the Fermat-Weber center $\mathcal{FW}_P$ of P is located at the origin (i.e. at point $(0,0)$). In order to compute μ_P^*, it is enough to compute the average distance between $\mathcal{FW}_P$ and the points in one of the four identical triangles forming P. We compute the average distance between $\mathcal{FW}_P$ and the points in the upper right triangle. The following expression corresponds to the sum of the distances from $\mathcal{FW}_P$ to the points in this triangle divided by the area of the triangle.

$$\lim_{\varepsilon \to 0} \frac{\int_0^1 (\int_0^{\varepsilon - \varepsilon x} \sqrt{x^2+y^2}\ dy)\ dx}{\frac{\varepsilon}{2}} = \frac{1}{3}$$

Since $\Delta(P) = 2$ we obtain that $\mu_P^* = \Delta(P)/6$ when ε tends to 0. ■

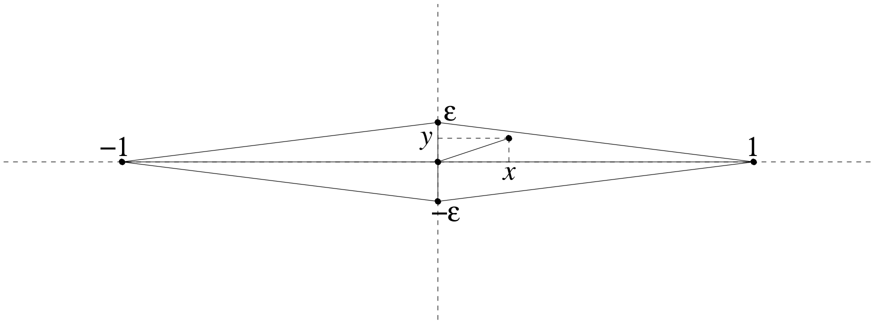

Figure 3: When ε tends to 0, $\mu_P^* = \Delta(P)/6$.

3 Approximation Algorithm

Given a convex polygon P and a parameter $\varepsilon > 0$, we show how to find a point $p \in P$ such that $\mu_P(p) \leq (1+\varepsilon)\mu_P^*$. Consider the grid G with side length $\delta\Delta(P)$, where $\delta = \varepsilon/25$. Let U be the set of all grid points that lie inside P. U induces a partition of P into $|U|$ regions; the region of P associated with $u \in U$, denoted r_u, is the intersection of P and the square of side length $\delta\Delta(P)$ centered at u. Put $w_u = \text{area}(r_u)$. For a point $y \in P$, let $\mu_U(y) = (\sum_{u \in U} \|yu\|\, w_u)/\text{area}(P)$. The following lemma tells us that $\mu_U(y)$ is a good approximation of $\mu_P(y)$.

Lemma 3.1 *For any point $y \in P$, we have $(1-\varepsilon/5)\mu_P(y) \leq \mu_U(y) \leq (1+\varepsilon/5)\mu_P(y)$.*

Proof: For a point $p \in P$, let $U(p)$ be the point $u \in U$ such that p lies in r_u. We prove the left inequality; the proof for the right inequality is very similar.

$$\int_{p \in P} \|yp\| \, dp \le \int_{p \in P} (\|yU(p)\| + \|U(p)p\|) dp \le \sum_{u \in U} \|yu\| \, w_u + \frac{\sqrt{2}}{2} \delta \Delta(P) \text{area}(P) \; .$$

Rearranging and diving by area(P), we obtain

$$\mu_P(y) - \frac{\sqrt{2}}{2} \delta \Delta(P) \le \mu_U(y) \; .$$

But, by the result of the previous section, the left side of the inequality above is greater or equal to

$$\mu_P(y) - \frac{\sqrt{2}}{2} 7 \delta \mu_P^* \ge (1 - \frac{\sqrt{2}}{2} 7 \delta) \mu_P(y) \ge (1 - \varepsilon/5) \mu_P(y) \; .$$

This concludes the proof of the left inequality. ■

Lemma 3.2 *Let u_0 be a point in U that minimizes $\mu_U(\cdot)$, then $\mu_P(u_0) \le (1+\varepsilon)\mu_P^*$.*

Proof: We first observe that $\mu_U(U(\mathcal{FW}_P)) \le (1+\varepsilon/5)\mu_P^*$, where $\mathcal{FW}_P$ is a Fermat-Weber center of P and $U(\mathcal{FW}_P)$ is the point in u whose associated region contains $\mathcal{FW}_P$. Indeed

$$\begin{aligned} \frac{\int_{p \in P} \|U(\mathcal{FW}_P)p\| \, dp}{\text{area}(P)} &\le \frac{\int_{p \in P} (\|U(\mathcal{FW}_P)\mathcal{FW}_P\| + \|\mathcal{FW}_P p\|) dp}{\text{area}(P)} \\ &\le \frac{\sqrt{2}}{2} \delta \Delta(P) + \mu_P^* \le (1 + \frac{\sqrt{2}}{2} 7 \delta) \mu_P^* \le (1 + \varepsilon/5) \mu_P^* \; . \end{aligned}$$

Thus, applying the lemma above

$$\mu_U(U(\mathcal{FW}_P)) \le (1+\varepsilon/5)\mu_P(U(\mathcal{FW}_P)) \le (1+\varepsilon/5)(1+\varepsilon/5)\mu_P^* \le (1 + 11\varepsilon/25)\mu_P^* \; .$$

Now, by the lemma above

$$\mu_P(u_0) \le \frac{\mu_U(u_0)}{1-\varepsilon/5} \le \frac{\mu_U(U(\mathcal{FW}_P))}{1-\varepsilon/5} \le \frac{(1+11\varepsilon/25)}{1-\varepsilon/5} \mu_P^* \le (1+\varepsilon)\mu_P^* \; .$$

■

Theorem 3.3 *Given a convex polygon P with n vertices and a parameter ε, one can compute in $O(n + 1/\varepsilon^4)$ time a point $p \in P$, such that $\mu_P(p) \le (1+\varepsilon)\mu_P^*$.*

Proof: We can compute the partition of P into cells induced by the grid G (by sweeping) in $O(n + 1/\varepsilon^2)$ time. At the end of the sweep we also have the set U of grid points inside P, and their respective weights. We now need to compute for each point $u \in U$, the number $\mu_U(u)$. Doing this in the naive way, would require $O(|U|^2) = O(1/\varepsilon^4)$ time. ■

In some cases it is possible to compute an approximate Fermat-Weber center point using the algorithm above (possibly with some modification), even if the underlying demand region P is not a convex polygon. One such case is that of non-convex fat polygons. A polygon P is *α-fat* if there exist a disk D_{in} contained in P and a disk D_{out} containing P, such that the ratio between the radii of D_{out} and D_{in} is at most α. Notice that the Fermat-Weber center of a non-convex polygon P is not necessarily in P, but it is clearly in any disk containing P.

Corollary 3.4 *Let P be a non-convex α-fat polygon. Then one can compute in polynomial time a point p, such that $\mu_P(p) \leq (1+\varepsilon)\mu_P^*$.*

Proof: Let D_{in} be a largest disk contained in P and let c_{in} be its center, and let D_{out} be a smallest disk containing P. Then

$$\mu_P^* \geq \int_{p \in D_{\text{in}}} \left\| p c_{\text{in}} \right\| dp / Area(D_{\text{out}}) \geq \Delta(D_{\text{in}})/(3\alpha^2) \geq \Delta(P)/(3\alpha^3) \ .$$

We can now apply the approximation algorithm described above with slight modification to find the desired point p. The main difference is that we need to consider all grid points in D_{out} as candidates, even if they lie outside P. ■

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