A Note of Fractional Independent Segments

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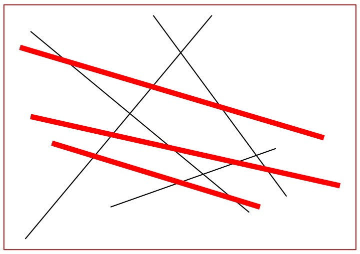
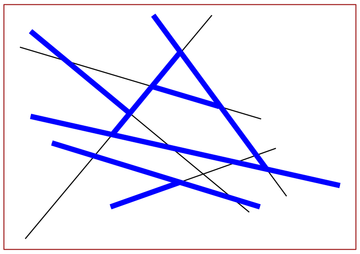
1 Introduction



Given a set $\mathcal{S} = \{\mathsf{s}_1, \ldots, \mathsf{s}_n\}$ of n segments in the plane, we are interested in finding an independent subset of segments $X \subseteq \mathcal{S}$ of maximum cardinality; that is, any distinct pair of segments X does not intersect. An input and a possible solution are depicted on the right. There is relatively little known about this problem. A (roughly) $O(\sqrt{\mathrm{opt}})$ approximation is known [AM06], but no matching lower bound is known.



Consider a relaxed version of the problem, referred to as the ***fractional independent segments problem***. Here, we are interested in picking a set of subsegments $\mathsf{t}_i \subseteq \mathsf{s}_i$, for $i = 1, \ldots, n$, such that $T = \{\mathsf{t}_1, \ldots, \mathsf{t}_n\}$ are interior disjoint, and the total length of these subsegments is maximized; that is the target is to maximize $|T| = \sum_{i=1}^{n} |\mathsf{t}_i|$, where $|\mathsf{t}_i|$ denotes the length of t_i. A possible solution is depicted on the right. Note, that it visually somewhat similar to the motorcycle graph [EE99]. The hope is that this problem is easier to solve than the original problem, and solving it would provide us with an insight to the original problem.

In this note, we show a randomized algorithm that outputs a solution of (expected) total length $\Omega(|\mathcal{S}| / \sqrt{n})$.

1.1 The algorithm

For $i = 1, \ldots, n$, we break s_i at each of its intersection with other segments of $\mathcal{S}$, and refer to the resulting segments as ***atomic***. Let L_i denote this set of atomic segments generated from s_i and $n_i = |L_i|$, for $i = 1, \ldots, n$. If $n_i \leq 4\sqrt{n}$, then we pick the longest segment in L_i to be t_i.

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Otherwise, let $f_i = \lceil n_i / 2\sqrt{n} \rceil$. We break s_i into groups of consecutive atomic intervals, where every such group is made out of at most $4f_i$ intervals, and s_i is covered by $\lceil \sqrt{n} \rceil$ such intervals. We randomly pick one of this chunks to be t_i, where all the $\lceil \sqrt{n} \rceil$ chunks have equal probability to be picked.

Next, we scan the resulting set of segments $T' = \{\mathsf{t}_1, \ldots, \mathsf{t}_n\}$, and we throw away segments from T' that intersect other segments in T' in an arbitrary fashion (here, both segment have to intersect in their interiors), till this set of segments become independent. Let T denote the resulting set of segments.

Lemma 1.1 *It holds that* $\mathbf{E}\Big[|T|\Big] = \Omega(|\mathcal{S}| / \sqrt{n})$.

Proof: First, observe that $\mathbf{E}[|\mathsf{t}_i|] \geq \Omega(|\mathsf{s}_i| / \sqrt{n})$, for $i = 1, \ldots, n$. Indeed, if $n_i \leq 4\sqrt{n}$, then this holds trivially, as we always pick the longest atomic segment of s_i to be t_i. Furthermore, since its atomic it can not intersect (in its interior) any other segment of T', and as such it is present in T.

Otherwise, every atomic segment of L_i has probability $1 / \lceil \sqrt{n} \rceil$ to be covered by t_i. As such, $\mathbf{E}[|\mathsf{t}_i|] = \sum_{\mathsf{u} \in L_i} |\mathsf{u}| / \lceil \sqrt{n} \rceil = \Omega(|\mathsf{s}_i| / \sqrt{n})$.

Finally, we claim that the probability of a segment t_i to survive the purge of T' used to create T is at least some fixed constant. Indeed, assume t_i was picked, and we now pick all the other intervals. Consider a vertex v formed by the intersection of s_i and s_j that is covered by t_i. Clearly, v has probability at most $p = 1 / \lceil \sqrt{n} \rceil$ to be covered by t_j. As such t_i survives if all the vertices it covers are not covered by their respective segments. The probabilities of the vertices covered by t_i to be covered (by their respective segment) are independent. Furthermore, the number of such segments is at most $\sqrt{n}$. and as such, we have that

$$
\begin{aligned}
\beta_i &= \mathbf{Pr}[\mathsf{t}_i \in T] \geq \mathbf{Pr}[v \in \mathsf{t}_i \text{ and } v \text{ is not covered otherwise}] \\
&\geq \prod_{v \in \mathsf{t}_i} (1 - p) \geq \left(1 - \frac{1}{\lceil \sqrt{n} \rceil}\right)^{\sqrt{n}} \geq \frac{1}{10}.
\end{aligned}
$$

We now have that

$$
\begin{aligned}
\mathbf{E}\Big[|T|\Big] &= \sum_i \mathbf{Pr}\big[\mathsf{t}_i \in T\big] \, \mathbf{E}\Big[|\mathsf{t}_i| \;\Big|\; \mathsf{t} \in T\Big] = \sum_i \beta_i \, \mathbf{E}\Big[|\mathsf{t}_i| \;\Big|\; \mathsf{t} \in T\Big] \\
&\geq \sum_i \frac{\mathbf{E}\big[|\mathsf{t}_i|\big]}{10} = \Omega\left(\sum_i \frac{\mathbf{E}\big[|\mathsf{s}_i|\big]}{\sqrt{n}}\right) = \Omega\left(\frac{|T|}{\sqrt{n}}\right),
\end{aligned}
$$

as claimed. ■

2 Conclusions

This seems to be a curios problem, and we leave it as an open problem to further improve it. Despite some effort I was unable to improve this result (which is somewhat weak and silly).

Acknowledgments

David Eppstein pointed out the similarity to the motorcycle graph.

References

[AM06] P. K. Agarwal and N. H. Mustafa. Independent set of intersection graphs of convex objects in 2D. *Comput. Geom. Theory Appl.*, 34(2):83–95, 2006.

[EE99] D. Eppstein and J. Erickson. Raising roofs, crashing cycles, and playing pool: Applications of a data structure for finding pairwise interactions. *Discrete & Computational Geometry*, 22(4):569–592, 1999.